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Percolation and Julia Sets

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The types of Julia sets for the renormalization group (RG) transformation for percolation on the hierarchial model are derived. The RG transformation for the concentration for two-dimensional triangular lattice site percolation problems and two-dimensional square lattice bond percolation problems are generalized to the complex plane. Julia sets for both cases are found.

1. INTRODUCTION

An interesting relation has been discovered (Derrida *et al.*, 1983) between the possible locations of phase transitions and the Julia set (Devany, 1989) of the renormalization group (RG) transformation. This relation has been used to study Ising and Potts models (Derrida *et al.*, 1983; Peitgen and Richter, 1984). However, it has not been used to study percolation (Stauffer and Aharony, 1992) explicitly. In the following we study some percolation RG transformations and determine the types of their Julia sets (Cantor type, closure of simply connected curves, or dendrite). In Section 2 we give a brief review of the definitions and the basic theorems of Julia and Fatou (Julia, 1918; Fatou, 1919). Some results which relate the Julia set of the map $z^2 + k$ to that of a class of rational maps are studied. In Section 3 the relation between phase transitions and Julia sets is reviewed. The RG transformation for percolation on the hierarchial lattice (Derrida *et al.*, 1983) is studied and it is shown that the corresponding Julia set is the union of simply connected curves. The RG transformations for the critical concentration for the two-dimensional triangular and square lattice are found and the corresponding Julia sets are the closures of simply connected curves.

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2. BASIC RESULTS ON JULIA SETS

Let $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, and $R(Z)$ be a rational function $R: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Let $R^n(z)$ be the nth iteration of R, i.e.,

$$R^n(z) = R(R(\cdots R(z)\cdots)) \tag{1}$$

The point z is a periodic point of $R(z)$ of period n if $R^n(z) = z$. The set of all such points is denoted by $\text{Per}_n(R)$. If $n = 1$, then z is a fixed point of R. The point z is a critical point of R if $R'(z) = 0$. Let z be a period-n point and let $\lambda = |R'(z)|$; then z is an attractor (repellor) if $\lambda < 1$ ($\lambda > 1$). If $\lambda = 1$, then z is a neutral point. A basin of attraction for an attractor of R, say z, is $\{z | R^k(z) \to z \text{ as } k \to \infty\}$. The Julia set for R, $J(R)$, is the closure of the set of repelling points of R.

Theorem 1 (Julia and Fatou). Julia sets have the following properties:

1. $J(R) \neq \varnothing$ (empty set).
2. $J(R)$ and $J(R^k)$ are identical, $k = 1, 2, \ldots$.
3. $J(R)$ is invariant under R.
4. For any $z \in J(R)$ then $\cup_{n=0}^{\infty} R^{-n}(z)$ is dense in $J(R)$.
5. $J(R)$ is equal to the boundary of the basin of attraction of attractive points of R.

Two maps f, g are called topologically conjugate if there is a homeomorphism h such that

$$g = h^{-1} \circ f \circ h \tag{2}$$

Theorem 2. If f and g are topologically conjugate rational maps on the Riemann sphere, then there is a one-to-one correspondence between their Julia sets if the inverse of the relating homeomorphism is also a homeomorphism.

Proof. Let $g = h^{-1} \circ f \circ h$ be topologically conjugate to f. For $z \in \text{Per}_n(g)$ then $z = g^n(z) = (h^{-1} \circ f \circ h)^n(z)$, i.e. $f^n(h(z)) = h(z)$. Therefore $z \in \text{Per}_n(g) \Rightarrow h(z) \in \text{Per}_n(f)$. The proof of the other direction follows by realizing that both h and h^{-1} are homeomorphisms. Thus there is a one-to-one correspondence between the sets of periodic points of period n for f and g. Let z be a repelling periodic n-point for

$$g(z) \Rightarrow |(g^n(z))'(z)| > 1 \Rightarrow |(h^{-1} \circ f \circ h)^n(z)| > 1$$

Using

$$(h^{-1} \circ f \circ h)'(z) = (h^{-1})'(f^n(z))[f^n(h(z))]'h'(z)$$

$$f^n(h(z)) = h(z)$$

and

$$(h^{-1}(h(z)))'h'(z) = 1$$

one gets $|(f^n(h(z)))'| > 1$, i.e., $h(z)$ is a repelling point for f. Consequently $z \in J(g) \Rightarrow h(z) \in J(f)$. Again the opposite direction is proved using the fact that h is a homeomorphism. ∎

Now it is straightforward to prove the following corollary:

Corollary 3. Julia set for $g(z)$, given by

$$g(z) = \frac{z^2[da^2 + c^2(dk - b)] + z[2abd + 2cd(dk - b)] + [db^2 + d^2(dk - b)]}{z^2[-ca^2 + c^2(a - ck)] + z[-2abc + 2cd(a - ck)] + [-cb^2 + d^2(a - ck)]} \tag{3}$$

where $(ad - bc) \neq 0$, is the image of $J(z^2 + k)$ under the Möbius transformation

$$h(z) = \frac{az + b}{cz + b} \tag{4}$$

One can get the solution of the Cayley problem that the Julia set for the map $\frac{1}{2}(z + 1/z)$ is the imaginary axis by choosing $a = b = c = 1$, $d = -1$, and $k = 0$ and by recalling that $J(z^2)$ is the unit circle.

3. PERCOLATION AND JULIA SETS

The idea of Yang and Lee (1952) is that the candidate points for phase transition occur at the zeros of the partition function of the system. Since it has no zeros for real values of the temperature, they looked for complex values of the temperature. In the thermodynamic limit these zeros may have a real limit which will correspond to the real critical temperature. This idea was exploited further by Derrida *et al.* (1983) to relate this algorithm to Julia sets, as follows: A typical renormalization group transformation reduces the number of particles N, but preserves the partition function (up to a constant)

$$\mathfrak{Z}_N(T) = \mathfrak{Z}_{N'}(T'), \qquad T' = R(T), \qquad N' < N \tag{5}$$

Let $\{\mathfrak{Z}_l\}$ $(\{\mathfrak{Z}_l'\})$ be the set of zeros for $\mathfrak{Z}_N$ $(\mathfrak{Z}_N')$; thus

$$\prod_l (\mathfrak{Z} - \mathfrak{Z}_l) = \prod_k (\mathfrak{Z}' - (\mathfrak{Z}_k')$$

Since $\mathfrak{Z}_N = 0 \Leftrightarrow \mathfrak{Z}_{N'} = 0$, one gets

$$\{\mathfrak{Z}_l\} = R^{-1}\{\mathfrak{Z}_k'\} \tag{6}$$

Therefore, to find the set of zeros for the partition function $\mathfrak{Z}_N$ one can begin with a simpler one, $\mathfrak{Z}_{N'}'$, $N' << N$, then use the inverse iteration (6). The complete set of zeros in the thermodynamic limit $N \to \infty$ is

$$\{\mathcal{Z}_l\} = \bigcup_{n=0}^{\infty} R^{-n}\{\mathcal{Z}'_k\} \tag{7}$$

Recalling property 4 in Section 2, it is clear that if R is a rational map, then the set of zeros of the partition function is the Julia set $J(R)$. The critical temperature (concentration) is the intersection of $J(R)$ with the real axis.

They applied their procedure to the q-state Potts model on the hierarchial lattice. However, they did not explicitly discuss percolation ($q = 1$). The RG transformation is

$$R(z) = \left(\frac{z^2}{2z - 1}\right)^2 \tag{8}$$

The set of fixed points is $\{0, 0.382, 1, 2.618\}$. The critical points are $z = 0, 1$, which are fixed points. Therefore the Julia set is the closure of an infinite number of simply connected curves.

Now we extend this construction to the RG transformations which relate the concentrations in site and pond percolation problems. For the triangular lattice site the RG transformation is

$$R(p) = p^3 + 3p^2(1 - p) \tag{9}$$

where p is the fraction of occupied sites. Generalizing p to the complex plane, one gets

$$R(z) = 3z^2 - 2z^2 \tag{10}$$

The set of fixed points is $\{0, 0.5, 1\}$, while the critical points are $\{0, 1\}$, therefore the corresponding Julia set is the closure of an infinite number of simply connected curves and is symmetric about the point (0.5, 0). Using

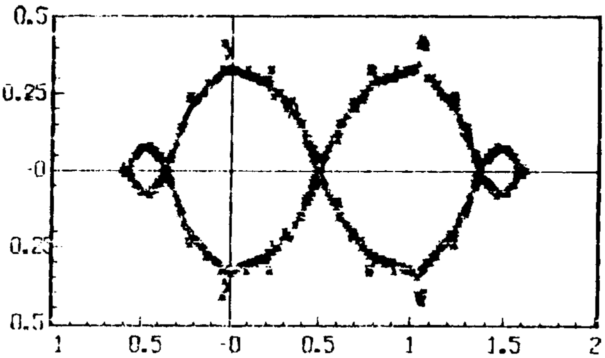

Fig. 1. Julia set corresponding to the transformation $R(z) = 3z^2 - 2z^3$.

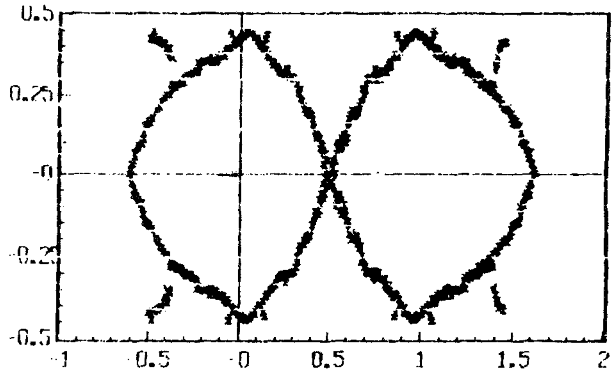

Fig. 2. Julia set corresponding to the transformation $R(z) = 2z^5 - 5z^4 + 2z^3 + 3z^2$.

property 5 in Section 2, we can generate the Julia set corresponding to the above transformation (see Fig. 1). Also for bond percolation problems corresponding to the square lattice the RG transformation is given by

$$R(z) = 2z^5 - 5z^4 + 2z^3 + 2z^2 \tag{11}$$

The set of fixed points is $\{0, 1, 0.5, 1.6180339, -0.6180339\}$ and the set of critical points is $\{0, 1\}$; therefore the corresponding Julia set is the closure of an infinite number of simply connected curves and is also symmetric about (0.5, 0) (see Fig. 2). The point, 1.6180339, −0.6180339 are 2-cycles.

4. CONCLUSION

We have derived the type of Julia set for the RG transformation for percolation on the hierarchial model. The RG transformation for the concentration for two-dimensional triangular lattice site percolation problems and two-dimensional square lattice bond percolation problems have been generalized to the complex plane. Julia sets for both cases have been found. These have been shown to be simply connected curves and symmetric about the point (0.5, 0).

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